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Bounds for the Norm of Certain Spline Projections, II*

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1. INTRODUCTION AND NOTATION

Let n and q be given natural numbers such that $n+1 \geqslant q > 0$ ($n > 0$). By I we denote the unit interval $[0, 1]$ and Δ_n is an arbitrary but fixed partition of the interval I:

$$\Delta_n : 0 = x_0 < x_1 < \cdots < x_n = 1.$$

By $\mathrm{Sp}(2q-1, \Delta_n)$ we denote the space of *polynomial splines of degree* $2q-1$, deficiency 1, and knots x_i ($i = 0, 1, \ldots, n$). Thus $s \in \mathrm{Sp}(2q-1, \Delta_n)$ if and only if

(i) in each interval $[x_{i-1}, x_i]$ ($i = 1, 2, \ldots, n$) s coincides with an algebraic polynomial of degree $2q-1$ or less,

(ii) $s \in C^{2q-2}(I)$.

It is known that $\mathrm{Sp}(2q-1, \Delta_n)$ is a linear subspace of the space $C(I)$ and $\dim \mathrm{Sp}(2q-1, \Delta_n) = n + 2q - 1$ (cf. [1]). In the sequel we will assume that each element s from the space $\mathrm{Sp}(2q-1, \Delta_n)$ satisfies additionally some boundary conditions

$$s^{(j)}(0) = s^{(j)}(1) = 0 \qquad (j = 1, 2, \ldots, q-1), \tag{1.1}$$

or

$$s^{(j)}(0) = s^{(j)}(1) = 0 \qquad (j = 2, 4, \ldots, 2q-2). \tag{1.2}$$

The conditions (1.2) are called Lidstone-type conditions (cf. [8]).

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It is known (see, e.g., [1]) that for given real numbers f_i $(i=0, 1, ..., n)$ there exists exactly one $s \in \mathrm{Sp}(2q-1, \Delta_n)$ interpolating the data f_i, i.e.,

$$s(x_i) = f_i \qquad (i=0, 1, ..., n), \tag{1.3}$$

jointly with the boundary conditions (1.1) or (1.2) (cf. [1]).

Every such spline function s may be written in the Lagrange form

$$s(x) = \sum_{i=0}^{n} f_i s_i(x) \qquad (x \in I),$$

where $s_i \in \mathrm{Sp}(2q-1, \Delta_n)$, $s_i(x_j) = \delta_{ij}$ $(i, j = 0, 1, ..., n)$ and every function s_i satisfies the boundary conditions (1.1) or (1.2). The functions s_i are the so-called *fundamental spline functions*. They play an important role in our further considerations. Consider the operator P_n^{2q-1} defined by

$$(P_n^{2q-1} f)(x) = \sum_{i=0}^{n} f(x_i)\, s_i(x) \qquad (f \in C(I)). \tag{1.4}$$

It is obvious that P_n^{2q-1} is a linear, bounded and idempotent map from $C(I)$ onto $\mathrm{Sp}(2q-1, \Delta_n)$; thus P_n^{2q-1} is a *projection*.

Let $\|\cdot\|_\infty$ stand for the sup-norm in the interval I. The inequality

$$\|f - P_n^{2q-1}\|_\infty \leqslant (1 + \|P_n^{2q-1}\|)\, \mathrm{dist}(f, \mathrm{Sp}(2q-1, \Delta_n))$$

is well known (here $f \in C(I)$). The operator norm $\|\cdot\|$ is defined in the usual way,

$$\|P_n^{2q-1}\| = \sup_{\|f\|_\infty \leqslant 1} \|P_n^{2q-1} f\|_\infty.$$

From this inequality we see that the knowledge on the size of the norm P_n^{2q-1} is important.

In this paper we will give some results concerning the norms of the projections P_n^{2q-1}. We continue our earlier investigations from [22], where the natural boundary conditions were imposed on the spline function $s \equiv P_n^{2q-1} f$. For other results for the non-periodic boundary conditions see [2–4, 12, 29]. In the case of the periodic boundary conditions (i.e., such that $s^{(j)}(0) = s^{(j)}(1)$ for $j = 0, 1, ..., 2q-2$) many results are known up to date (see [6, 12–20, 24–28]).

In Section 3 the cubic case $(q=2)$ is treated assuming the boundary conditions (1.1). For the second type boundary conditions some results are given in the above-mentioned paper [22]. Estimations from above for $\|P_n^3\|$ (for arbitrary knots) and explicit formulae for these norms for equidistant knots are given. In the final section the uniform upper bounds for $\|P_n^5\|$ are derived (in the case of equidistant knots). The interpolant $P_n^5 f$ satisfies the boundary conditions (1.1) or (1.2).

For the related results concerning the norm of some quadratic spline projections, see, [3, 7, 10, 11, 19, 20, 23–25, 29].

2. LEMMAS

For our further aims we define the bi-infinity sequence $\{d_i\}$ in the following manner: $d_{-i}=0$, $d_0=1$, $d_1=4$, $d_{i+1}=4\,d_i-d_{i-1}$ $(i=1,2,\ldots)$.

LEMMA 2.1. *If the numbers d_i are defined as above, then*

$$\begin{aligned} d_i d_l - d_{i-1}d_{l+1} &= d_{l-i} && \text{if } 0\leqslant i\leqslant l+1,\\ &= -d_{i-l-2} && \text{if } i\geqslant l+1. \end{aligned}$$

Proof. Since $d_m=[(2+(3)^{1/2})^{m+1}-(2-(3)^{1/2})^{m+1}]/(2(3)^{1/2})$ $(m=-1,0,\ldots)$, then the desired result follows by direct calculations. ∎

Let $\beta_{j,-1}=\beta_{j0}=\beta_{jn}=\beta_{j,n+1}=0$, and

$$\begin{aligned} \beta_{ij} &= (-1)^{i+j}\,d_{j-1}d_{n-i-1}/d_{n-1} && (j\leqslant i),\\ &= (-1)^{i+j}\,d_{i-1}d_{n-j-1}/d_{n-1} && (j\geqslant i) \end{aligned}$$
$$(i,j=1,2,\ldots,n-1). \tag{2.1}$$

We have the following

LEMMA 2.2. *If the numbers $m_j^{(i)}$ are such that*

$$\begin{aligned} m_{j-1}^{(i)}+4m_j^{(i)}+m_{j+1}^{(i)} &= 3n(\delta_{j+1,i}-\delta_{j-1,i}),\\ m_0^{(i)}=m_n^{(i)}=0 \qquad & (i=0,1,\ldots,n;\ j=1,2,\ldots,n-1), \end{aligned} \tag{2.2}$$

then

$$\begin{aligned} m_j^{(i)} &= (-1)^{i+j+1}\,3n\,d_{j-1}(d_{n-i}-d_{n-i-2})/d_{n-1} && (j<i),\\ &= 3n(d_{i-1}d_{n-i-2}-d_{i-2}d_{n-i-1})/d_{n-1} && (j=i),\\ &= (-1)^{i+j}\,3n\,d_{n-j-1}(d_i-d_{i-2})/d_{n-1} && (j>i) \end{aligned}$$
$$(i=0,1,\ldots,n;\ j=1,2,\ldots,n-1). \tag{2.3}$$

Proof. It is known (see, e.g., [21]) that the matrix of the above linear system (2.2) possesses an inverse with entries β_{ij} given by (2.1). By virtue of (2.2) we have

$$m_j^{(i)} = 3n(\beta_{j,i-1} - \beta_{j,i+1}).$$

Hence and from (2.1) we obtain the desired result (2.3). ∎

LEMMA 2.3. *Let* $x_i = i/n$, $s_i \in \mathrm{Sp}(3, \Delta_n)$ $(i = 0, 1,..., n)$ *and let each fundamental spline function* s_i *satisfy the boundary conditions* (1.1) *for* $q = 2$. *If* $x \in (x_{j-1}, x_j)$ $(j = 1, 2,..., n)$, *then*

$$\begin{aligned} \operatorname{sgn} s_i(x) &= (-1)^{i+j} \qquad (j \leqslant i), \\ &= (-1)^{i+j+1} \qquad (j > i) \end{aligned}$$
$$(i = 0, 1,..., n; j = 1, 2,..., n). \tag{2.4}$$

The proof of (2.4) follows immediately from Theorem 2 (Part I) in [5]. ∎

3. Cubic Case

For the sake of brevity we introduce more notation. Let $h_j = x_j - x_{j-1}$ $(j = 1, 2,..., n)$, $h = \max_{1 < j < n} h_j$, $\alpha = \max_{|i-j|=1} h_i/[h_j(h_i + h_j)]$.

Our first result is contained in the following

THEOREM 3.1. *Let the knots* x_i *be arbitrary,* $(P_n^3 f)(x_i) = f(x_i)$ $(i = 0, 1,..., n)$ *and* $(P_n^3 f)'(0) = (P_n^3 f)'(1) = 0$. *Then*

$$\|P_n^3\| \leqslant 1 + \tfrac{3}{2}\alpha h, \tag{3.1}$$

where α *and* h *are defined as above.*

The proof is quite similar to that of [6, Theorem 1]. For this reason it is omitted. ∎

From (3.1) we have the following

COROLLARY 3.1. *For equidistant knots we have* $\|P_n^3\| \leqslant \frac{7}{4}$.

Now we shall give an explicit formula for the norm of the projection P_n^3 in the case of equidistant knots. Let

$$\Lambda_n^{2q-1}(x) = \sum_{i=0}^{n} |s_i(x)| \qquad (x \in I),$$

denote the so-called *Lebesgue function* for the projection P_n^{2q-1}. It is known that $\|P_n^{2q-1}\| = \|\Lambda_n^{2q-1}\|_\infty$. From this equality it follows that for our aims we must have more information on the functions s_l. Let $m_i^{(l)} = s_l'(x_i)$ $(i, l = 0, 1, \ldots, n)$. By virtue of our assumptions we have $m_0^{(l)} = m_n^{(l)} = 0$ for all l. If $x \in [x_{i-1}, x_i]$ $(i = 1, 2, \ldots, n)$ and if knots x_i are equidistant, then each fundamental spline $s_l(x)$ may be written as

$$s_l(x) = \delta_{i-1,l}\Phi_0(x) + \delta_{il}\Phi_0(1-x) + m_{i-1}^{(l)}\Phi_1(x) - m_i^{(l)}\Phi_1(1-x)$$
$$(l = 0, 1, \ldots, n; x \in [x_{i-1}, x_i]; i = 1, 2, \ldots, n-1), \tag{3.2}$$

where

$$\begin{aligned} \Phi_0(x) &= (1+2t)(1-t)^2, \\ \Phi_1(x) &= t(1-t)^2/n, \qquad t = n(x - x_{i-1}), \end{aligned} \tag{3.3}$$

(see, e.g., [1, 6]). If $x \in [x_{i-1}, x_i]$, then $\Phi_0(x)$, $\Phi_0(1-x)$, $\Phi_1(x)$, $\Phi_1(1-x) \geqslant 0$, and

$$\Phi_0(x) + \Phi_0(1-x) = 1,$$
$$\Phi_1(x) + \Phi_1(1-x) \leqslant 1/4n.$$

With the help of Lemma 2.3 and (3.2) one can prove

$$\Lambda_n^3(x) = 1 + \alpha_{i,n}\Phi_1(x) - \beta_{i,n}\Phi_1(1-x)$$
$$(x \in [x_{i-1}, x_i]; i = 1, 2, \ldots, n),$$

where

$$\alpha_{i,n} = \sum_{l=0}^{i-1} (-1)^{i+l+1} m_{i-1}^{(l)} + \sum_{l=i}^{n} (-1)^{i+l} m_{i-1}^{(l)},$$
$$\beta_{i,n} = \sum_{l=0}^{i-1} (-1)^{i+l+1} m_i^{(l)} + \sum_{l=i}^{n} (-1)^{i+l} m_i^{(l)}.$$

By virtue of (2.3) the above formulae simplify to

$$\begin{aligned} \alpha_{i,n} &= \frac{6}{d_{n-1}} d_{i-2}(d_{n-i-1} + d_{n-i}), \\ \beta_{i,n} &= -\frac{6}{d_{n-1}} d_{n-i-1}(d_{i-2} + d_{i-1}) \qquad (i = 1, 2, \ldots, n). \end{aligned} \tag{3.4}$$

Thus if $x \in [x_{i-1}, x_i]$, then the Lebesgue function $\Lambda_n^3(x)$ may be written as

$$\Lambda_n^3(x) = 1 + \frac{6}{d_{n-1}} [a_i(1-t) + b_i t]\, t(1-t) \qquad (t = n(x - x_{i-1});\ i = 1, 2, \ldots, n), \tag{3.5}$$

where

$$\begin{aligned} a_i &= d_{i-2}(d_{n-i-1} + d_{n-i}), \\ b_i &= d_{n-i-1}(d_{i-2} + d_{i-1}) \qquad (i = 1, 2, \ldots, n). \end{aligned} \tag{3.6}$$

Let $\lambda_i = \max_{x_{i-1} \leqslant x \leqslant x_i} \Lambda_n^3(x)$ $(i = 1, 2, \ldots, n)$. From (3.5) and (3.6) we see that $\Lambda_n^3(x) = \Lambda_n^3(1-x)$. Hence $\lambda_i = \lambda_{n+1-i}$ $(i = 1, 2, \ldots, n)$.

THEOREM 3.2. *Let* $x_i = i/n$, $(P_n^3 f)(x_i) = f(x_i)$ $(i = 0, 1, \ldots, n)$, $(P_n^3 f)'(0) = (P_n^3 f)'(1) = 0$. *Then*

$$\begin{aligned} \|P_n^3\| &= 1 + \frac{3}{2d_{n-1}} d_{j-1}(d_{j-1} + d_j) && \text{if } n = 2j+1 \quad (j = 0, 1, \ldots), \\ &= 1 + \frac{2}{9d_{n-1}} [2\delta^{3/2} + (3-\delta)(2a_j + 1)] && \text{if } n = 2j \quad (j = 1, 2, \ldots), \end{aligned}$$

where a_j *is defined in* (3.6) *and* $\delta = a_j^2 + a_j + 1$. *Moreover,*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{j+1};\quad \lambda_{j+1} > \lambda_{j+2} > \cdots > \lambda_n \qquad \text{if} \quad n = 2j+1,$$

and

$$\lambda_1 < \lambda_2 < \cdots < \lambda_j = \lambda_{j+1};\quad \lambda_{j+1} > \lambda_{j+2} > \cdots > \lambda_n \qquad \text{if} \quad n = 2j.$$

Proof. According to Lemma 2.1 we have, by virtue of (3.6),

$$\begin{aligned} a_{i+1} - a_i &= d_{n-2i-1} + d_{n-2i} && \text{if} \quad 2i \leqslant n, \\ &= -(d_{2i-n-1} + d_{2i-n-2}) && \text{if} \quad 2i + 1 \geqslant n. \end{aligned} \tag{3.7}$$

Now we consider two cases. Let

1°. $n = 2j + 1$ $(j = 0, 1, \ldots)$. By virtue of (3.6) and (3.7) one gets

$$\begin{aligned} a_{i+1} - a_i &> 0 && \text{if} \quad i = 1, 2, \ldots, j, \\ &< 0 && \text{if} \quad i = j+1, j+2, \ldots, n, \end{aligned}$$

and

$$\begin{aligned} b_{i+1} - b_i &> 0 \qquad \text{if} \quad i = 1, 2, \ldots, j, \\ &< 0 \qquad \text{if} \quad i = j+1, j+2, \ldots, n. \end{aligned}$$

Thus

$$a_1 < a_2 < \cdots < a_{j+1}; \quad a_{j+1} > a_{j+2} > \cdots > a_n,$$

and (3.8)

$$b_1 < b_2 < \cdots < b_{j+1}; \quad b_{j+1} > b_{j+2} > \cdots > b_n.$$

Moreover, $a_{j+1} = b_{j+1}$.

By virtue of (3.5) and (3.8) we have

$$\|P_n^3\| = \max_{x_j \leqslant x \leqslant x_{j+1}} \Lambda_n^3(x) = \Lambda_n^3\left(\frac{1}{2}\right) = 1 + \frac{3}{2d_{n-1}} d_{j-1}(d_{j-1} + d_j).$$

2°. $n = 2j$ ($j = 1, 2, \ldots$). Similarly to the previous case we can prove

$$a_1 < a_2 < \cdots < a_{j+1}; \quad a_{j+1} > a_{j+2} > \cdots > a_n,$$

and (3.9)

$$b_1 < b_2 < \cdots < b_j; \quad b_j > b_{j+1} > \cdots > b_n.$$

Moreover, $a_j = b_{j+1}$ and $a_{j+1} = b_j$. Hence $\|P_n^3\| = \max_{x_{j-1} \leqslant x \leqslant x_j} \Lambda_n^3(x) = \Lambda_n^3(t^*)$. If $x \in [x_{j-1}, x_j]$ and $n = 2j$, then, from (3.6) and (3.5), one has

$$\Lambda_n^3(x) = 1 + \frac{6}{d_{n-1}} [a_j + (d_{j-1}^2 - d_{j-2} d_j)\, t]\, t(1-t). \qquad (3.10)$$

From Lemma 2.1 one gets $d_{j-1}^2 - d_{j-2} d_j = 1$. The cubic polynomial $(a_j + t)\, t(1-t)$ attains its single maximum in the interval $[0, 1]$ in the point t^*, where

$$t^* = (\sqrt{\delta} - a_j + 1)/3 \qquad (1/3 < t^* < 1/2),$$

and δ is the same as above. With the help of (3.10) we obtain the desired result.

The last statement of the thesis follows immediately from (3.5), (3.8) and (3.9). ∎

In Table I we give values of $\|P_n^3\|$ and $e_n := \max_{1 \leqslant i \leqslant n} \lambda_i - \min_{1 \leqslant i \leqslant n} \lambda_i$ for small values of n.

COROLLARY 3.2. *If P_n^3 is defined as in Theorem 3.2, then*

$$\|P_1^3\| < \|P_3^3\| < \|P_5^3\| < \cdots < (1 + 3(3)^{1/2})/4 = 1.549038.... \quad ∎$$

TABLE I

n	$\|P_n^3\|$	e_n
1	1.0	0.0
2	1.222222	0.0
3	1.5	0.262963
4	1.522407	0.284312
5	1.545455	0.307284
6	1.547116	0.308939
7	1.548780	0.310603
8	1.548900	0.310723
9	1.549020	0.310843
10	1.549028	0.310851

4. Quintic Case

In this section we assume that the knots x_i are equidistant, i.e., $x_i = i/n$ for all $i = 0, 1,..., n$. We give below an upper bound for the norm of the projection P_n^5, under the assumption that the spline function $s = P_n^5 f$ satisfies the boundary conditions (1.1) or (1.2) for $q = 3$. Let us denote $f_j = s(x_j)$, $m_j = s'(x_j)$, $M_j = s''(x_j)$, $S_j = s^{\mathrm{IV}}(x_j)$ $(j = 0, 1,..., n)$. The first theorem follows.

THEOREM 4.1. *Let* $x_i = i/n$, $(P_n^5 f)(x_i) = f(x_i)$ $(i = 0, 1,..., n;\ f \in C(I))$ *and let* $(P_n^5 f)^{(j)}(0) = (P_n^5 f)^{(j)}(1) = 0$ *for* $j = 1, 2$. *Then*

$$\|P_n^5\| \leqslant 18\frac{73}{96}.$$

Proof. It is known that the first derivatives m_j satisfy the following consistency relations:

$$227m_1 + 79m_2 + 3m_3 = n(-235f_0 + 65f_1 + 155f_2 + 15f_3),$$

$$m_{j-2} + 26m_{j-1} + 66m_j + 26m_{j+1} + m_{j+2}$$
$$= 5n(-f_{j-2} - 10f_{j-1} + 10f_{j+1} + f_{j+2}) \qquad (j = 2, 3,..., n-2),$$

$$3m_{n-3} + 79m_{n-2} + 227m_{n-1} = n(-15f_{n-3} - 155f_{n-2} - 65f_{n-1} + 235f_n)$$

(see, e.g., [9]). Let A denote the matrix of the above system of linear equations with unknowns m_j $(j = 1, 2,..., n-1;\ m_0 = m_n = 0)$. Using the standard diagonal dominance argument we obtain $\|A^{-1}\|_\infty \leqslant 1/12$ (here $\|\cdot\|_\infty$ stands for the infinity norm of the square matrix). Now we take a function $f \in C(I)$ such that $\|f\|_\infty \leqslant 1$. Let $b = (b_1, b_2,..., b_{n-1})^T$, where b_j

denotes the right-hand side in the jth equation of the above system. It is obvious that

$$\max_{1 \leqslant j \leqslant n-1} |b_j| \leqslant 470n.$$

Hence

$$\max_{0 \leqslant j \leqslant n} |m_j| \leqslant \frac{235}{6} n. \tag{4.1}$$

Hoskins [9] proved that

$$M_{j-1} - 6M_j + M_{j+1} = -20n^2(f_{j-1} - 2f_j + f_{j+1}) + 8n(m_{j+1} - m_{j-1})$$
$$(j = 1, 2, ..., n-1; M_0 = M_n = 0).$$

Similarly to that above one can prove

$$\max_{0 \leqslant j \leqslant n} |M_j| \leqslant \frac{530}{3} n^2. \tag{4.2}$$

For $x \in [x_{i-1}, x_i]$ $(i = 1, 2, ..., n)$ the quintic spline $P_n^5 f$ may by written as

$$(P_n^5 f)(x) = f_{i-1} \Phi_0(t) + f_i \Phi_0(1-t) + [m_{i-1} \Phi_1(t) - m_i \Phi_1(1-t)]/n$$
$$+ [M_{i-1} \Phi_2(t) + M_i \Phi_2(1-t)]/n^2, \tag{4.3}$$

where $t = n(x - x_{i-1})$,

$$\begin{aligned} \Phi_0(x) &= (1-x)^3(1+3x+6x^2), \\ \Phi_1(x) &= x(1-x)^3(1+3x), \\ \Phi_2(x) &= x^2(1-x)^3. \end{aligned} \tag{4.4}$$

From (4.4) we see that $\Phi_i(x)$, $\Phi_i(1-x) \geqslant 0$ for $0 \leqslant x \leqslant 1$ and $i = 0, 1, 2$. We also have

$$\begin{aligned} &\Phi_0(x) + \Phi_0(1-x) = 1, \\ &\Phi_1(x) + \Phi_1(1-x) \leqslant \frac{5}{16}, \\ &\Phi_2(x) + \Phi_2(1-x) \leqslant \frac{1}{32} \qquad (0 \leqslant x \leqslant 1). \end{aligned} \tag{4.5}$$

Taking $f \in C(I)$ and such that $\|f\|_\infty \leqslant 1$ we obtain, by virtue of (4.1)–(4.3) and (4.5),

$$|(P_n^5 f)(x)| \leqslant 18 \frac{73}{96}.$$

Hence the desired result follows. ∎

In the case when the boundary conditions (1.2) are imposed on the spline function $P_n^5 f$ then the upper bound for the norm of this projection is given in the following

THEOREM 4.2. *Let* $x_i = i/n$, $(P_n^5 f)(x_i) = f(x_i)$ $(i = 0, 1, \ldots, n; f \in C(I))$ *and let* $(P_n^5 f)^{(j)}(0) = (P_n^5 f)^{(j)}(1) = 0$ *for* $j = 2, 4$. *Then*

$$\|P_n^5\| \leqslant \frac{21}{4}.$$

Proof. We only sketch the proof because it is quite similar to the proof of Theorem 4.1. The consistency relations for the fourth-order derivatives $S_j = s^{\mathrm{IV}}(x_j)$ are

$$65S_1 + 26S_2 + S_3 = 120n^4(-2f_0 + 5f_1 - 4f_2 + f_3),$$

$$S_{j-2} + 26S_{j-1} + 66S_j + 26S_{j+1} + S_{j+2}$$
$$= 120n^4(f_{j-2} - 4f_{j-1} + 6f_j - 4f_{j+1} + f_{j+2}) \qquad (j = 2, 3, \ldots, n-2),$$

$$S_{n-3} + 26S_{n-2} + 65S_{n-1} = 120n^4(f_{n-3} - 4f_{n-2} + 5f_{n-1} - 2f_n)$$

(see, e.g., [30]). Hence if $\|f\|_\infty \leqslant 1$ then

$$\max_{0 \leqslant j \leqslant n} |S_j| \leqslant 160n^4. \tag{4.6}$$

We also have $M_j = n^2(f_{j-1} - 2f_j + f_{j+1}) - (S_{j-1} + 8S_j + S_{j+1})/120n^2$ (see, e.g., [30]). By virtue of (4.6) one gets

$$\max_{0 \leqslant j \leqslant n} |M_j| \leqslant \frac{52}{3} n^2. \tag{4.7}$$

For $x \in [x_{i-1}, x_i]$ $(i = 1, 2, \ldots, n)$ the quintic spline $(P_n^5 f)(x)$ may be written as

$$(P_n^5 f)(x) = f_{i-1} \Psi_0(1-t) + f_i \Psi_0(t) + [M_{i-1} \Psi_1(1-t) + M_i \Psi_1(t)]/n^2$$
$$+ [S_{i-1} \Psi_2(1-t) + S_i \Psi_2(t)]/n^4, \tag{4.8}$$

where $t = n(x - x_{i-1})$,

$$\Psi_0(x) = x,$$
$$\Psi_1(x) = (x^3 - x)/6,$$
$$\Psi_2(x) = (x^5 - x)/120 - \Psi_1(x)/6.$$

We see that

$$\Psi_0(x) + \Psi_0(1-x) = 1,$$
$$|\Psi_1(x) + \Psi_1(1-x)| \leqslant \frac{1}{8},$$
$$\Psi_2(x) + \Psi_2(1-x) \leqslant \frac{5}{384}.$$

Hence and from (4.6)–(4.8) we obtain the desired result. ∎

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